JOURNAL OF APPROXIMATION THEORY 15, 275–277 (1975)

Uniqueness in One-Sided Linear Chebyshev Approximation

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario, London, Ontario, Canada

Communicated by Oved Shisha

Let X be a compact normal space and $C(X)$ the space of continuous functions on X. For $g \in C(X)$ define

$$\| g \| = \sup\{| g(x)|: x \in X\}.$$

Let $\{\phi_1, ..., \phi_n\}$ be a linearly independent subset of $C(X)$ and define

$$L(A, x) = \sum_{k=1}^{n} a_k \phi_k(x).$$

The problem of one-sided approximation is: for a given $f \in C(X)$ to find $L(A^*, .)$ minimizing $\| f - L(A, .)\|$ over all A with $L(A, .) \geqslant f$. Such an element $L(A^*, .)$ is called a best one-sided approximation to f.

We wish to find a necessary and sufficient condition for each $f \in C(X)$ to have a unique best approximation. It turns out that this condition is the classical Haar condition for uniqueness in ordinary Chebyshev approximation.

THEOREM. *A necessary and sufficient condition that each $f \in C(X)$ have a unique best approximation is that $\{\phi_1, ..., \phi_n\}$ be a Chebyshev set.*

Proof. Sufficiency. That $\{\phi_1, ..., \phi_n\}$ is a Chebyshev set implies that there is A with $L(A, .) > 0$. $L(A, .) > 0$ implies that for a given f there exists a μ such that $L(\mu A, .) > f$, and standard arguments then show the existence of a best approximation. Uniqueness is easily shown from known results [1; 2, pp. 446–450].

Necessity. If no $L(A, .) > 0$ exists, $f > 0$ has no best approximation. We will suppose that there is $L(A, .) > 0$. By arguments given above this implies the existence of a best approximation to all $f \in C(X)$. Let V_n be the linear space generated by $\{\phi_1, ..., \phi_n\}$. Suppose there exists $h_0 \in V_n$ which has distinct zeros $x_1, ..., x_n$ in X, then we shall show that there exists $f \in C(X)$ with more than one best approximation. Without loss of generality we can assume that $\| h_0 \| = 1$. By matrix theory it follows that we can find $B_1, \ldots, B_n$ not all zero such that

$$\sum_{i=1}^{n} B_i h(x_i) = 0 \qquad \textit{for every } h \in V_n .$$

We can assume without loss of generality that $B_j < 0$ for at least one j, $1 \leqslant j \leqslant n$. Let S be a continuous function on X such that $0 \leqslant S(x) \leqslant 1$ for all $x \in X$ and for $i = 1, \ldots, n$,

$$\begin{aligned} S(x_i) &= 0 \qquad \text{if} \quad B_i \geqslant 0, \\ &= 1 \qquad \text{if} \quad B_i < 0. \end{aligned}$$

Such a continuous function on X exists by Urysohn's lemma. Let

$$f(x) = \min\{-S(x)(1 - | h_0(x)|), h_0(x)\}$$

and let

$$\delta_n(f) = \inf\{\| f - h \|: h \in V_n \text{ and } f \leqslant h\}.$$

We shall show first that $\delta_n(f) \geqslant 1$. We have, for every $h \in V_n$, $f \leqslant h$,

$$\begin{aligned} \sum_{B_j<0} | B_j | &= \sum_{j=1}^{n} B_j f(x_j) \\ &= \sum_{j=1}^{n} B_j(f(x_j) - h(x_j)) \\ &= -\sum_{B_j<0} B_j(h(x_j) - f(x_j)) - \sum_{B_j\geqslant 0} B_j(h(x_j) - f(x_j)) \\ &\leqslant \delta_n(f) \sum_{B_j<0} | B_j |. \end{aligned}$$

This shows that $\delta_n(f) \geqslant 1$.

Next, from the definition of f it follows immediately that $\|f\| = 1$ and $\|f - h_0\| = 1$.

Since $f \leqslant 0$ and $f \leqslant h_0$, any one of these two relations implies that $\delta_n(f) \leqslant 1$. Hence, we have $\delta_n(f) = 1$.

Finally, since $f \leqslant 0$ and $f \leqslant h_0$ and $\|f - 0\| = \delta_n(f)$ and $\|f - h_0\| = \delta_n(f)$, the function f has two distinct best one-sided approximations from V_n, namely 0 and h_0, and the theorem is completely proved.

The proof of necessity parallels that of Meinardus [3, pp. 17–18]. It was suggested by the referee as more appropriate than the original proof of the author.

REFERENCES

1. C. B. DUNHAM, Chebyshev approximation with restricted ranges by families with the betweeness property, *J. Approximation Theory* **11** (1974), 254–259.
2. D. G. MOURSUND, Chebyshev approximation using a generalized weight function, *J. SIAM Numer. Anal.* **3** (1966), 435–450.
3. G. MEINARDUS, "Approximation of Functions: Theory and Numerical Methods," Springer–Verlag, New York, 1967.